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Environmental Recording of the Angular Momentum in Quantum Mechanics

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We discuss the coupling of a quantum system through the angular momentum to the reservoir of quantum harmonic oscillators. In classical mechanics an observation of the oscillator trajectories allows one to determine the system's angular momentum. We discuss the quantum dynamics of the model. We show that the model of an observation of environmental coordinates can be related to some models of angular momentum measurement based on a stochastic Schrödinger equation.

1. INTRODUCTION

A measurement in classical physics can be defined as an irreversible process (a record of a measuring device) which allows one to deduce the state of the system. It is believed that the disturbance caused by the apparatus can be arbitrarily small. In principle, we could describe the measurement in terms of Hamiltonian dynamics through a coupling of the system to an environment. The irreversibility results from an infinite number of degrees of freedom of the environment when most of them remain unobserved. Only few (macroscopic) parameters of the environment are recorded. Assume we wish to follow the above argument on deducing the system's dynamics from a state of a measuring device in the framework of quantum Hamiltonian dynamics. The first difficulty we encounter is the lack of a notion of a classical object in the framework of standard quantum mechanics. It is believed that only a state of a classical body can be directly accessible to an observation. Instead of considering a classical object in quantum mechanics, we suggest coupling a quantum system to the environment whose classical limit is well understood. In such a case the effect of the system upon the environment

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will be well under control both in standard quantum mechanics as well as in classical mechanics, where the notion of a measurement poses no difficulty. This way we gain some information about the state of a quantum system through a classical measurement.

We believe that the classical limit of the harmonic oscillator is quite well understood (e.g., through the construction of the coherent states). Then, we can describe light as a collection of harmonic oscillators. Subsequently, scattering of light on a quantum system gives information about the system which can be considered as a measurement. We think that the measurement of the properties of radiation is sufficiently understood in classical as well as quantum physics and that the relation between quantum measurement and classical measurement is quite clear in this case. For this reason we investigate here a model of an interaction of a quantum system with an environment consisting of an infinite set of oscillators [for a discussion of the relevance of the quantum environment to classical interpretation see Leggett *et al.* (1987)]. We investigate what kind of information about the dynamics of the system we gain through an observation of the oscillators (i.e., of the electromagnetic field). We are interested in a measurement of the angular momentum. The coupling of the angular momentum to the harmonic oscillators is defined in Section 2. We investigate in this section its classical and quantum dynamics (the Heisenberg picture). In Section 3 we describe the dynamics (the Schrödinger picture) in the stochastic formulation (Haba, 1994). We assume the conventional Born interpretation of the wave function as the sole input of the quantum measurement theory. We show that some models of state vector reduction based on a stochastic Schrödinger equation follow from the stochastic formulation of the standard quantum theory of an interaction of a system with an environment. If the measurement of oscillator positions is nonselective (i.e., results of the measurement are unknown), then we obtain a stochastic equation for the system's state vector and a master equation of Gisin (1984, 1989), Ghirardi *et al.* (1990), Gisin and Cibils (1992), Amman (1994), and Belavkin and Staszewski (1992). It follows from the master equation that in the mixed state the off-diagonal matrix elements in the basis of the measured M_3 angular momentum fall off exponentially fast. In this paper we restrict ourselves to a nonselective position measurement. We indicate how the results would be modified if the measurement was selective. We suggest also a method to treat a simultaneous approximate measurement of position and momentum.

A master equation describing the measurement of position has been obtained in Barchielli *et al.* (1982) and Caves and Milburn (1987) under the assumption of a continuous observation (the momenta of the meter could be considered as the environment in that case). An analogous equation for a continuous measurement of the angular momentum has been derived in

Sanders and Milburn (1989) [an outline of another derivation of such an equation appeared in Barchielli *et al.* (1982)].

2. THE MODEL AND ITS DYNAMICS IN THE HEISENBERG PICTURE

We couple a general three-dimensional Hamiltonian system H_S to an environment consisting of harmonic oscillators

$$H = \frac{1}{2m}\mathbf{P}^2 + V(\mathbf{X}) + \sum_k \frac{1}{2\mu_k} p_k^2 + \frac{1}{2}\mu_k\omega_k^2 x_k^2 + \sum_{\alpha=1}^{3} F_\alpha(x) M_\alpha \tag{1}$$

where $M_\alpha = \epsilon_{\alpha\beta\gamma} X_\beta P_\gamma$ (we shall denote coordinates of the system by capital letters) and the function F_α of oscillator coordinates will be specified later.

We can obtain models of this type from QED when a quantum particle interacts with a quantum electromagnetic field C_α in a finite volume Ω. Let us expand the vector potential C_α in eigenmodes $g(k, \mathbf{X})$:

$$\omega_k^2 g(k, \mathbf{X}) = \Delta g(k, \mathbf{X})$$

In this formula ω_k denotes the eigenfrequencies of the Laplacian Δ and k the eigenmodes (so k is a vector if Ω is a parallelepiped). We assume that $g(k, \mathbf{X})$ are real functions normalized as follows:

$$\int_\Omega g(k, \mathbf{X}) g(k', \mathbf{X})\, dX = \delta(k, k')\Omega$$

The δ-function is equal to 1 for the same g and equal to zero if the eigenmodes are different. The expansion in eigenmodes takes the form

$$C_\alpha(\mathbf{X}) = \sum_{k,\nu} \sqrt{\frac{2\pi\hbar}{\omega_k \Omega}} f(k) \mathscr{E}_\alpha(k, \nu)(a(k, \nu) + a(k, \nu)^+) g(k, \mathbf{X})$$

where $\mathscr{E}$ is the polarization transverse to k, $a(k, \nu)$ and $a(k, \nu)^+$ are the amplitudes fulfilling the (Poisson brackets) commutation relations for creation and annihilation operators, and $f(k)$ is a form factor regularizing the electromagnetic field. The amplitude can be expressed by the position and momentum of an oscillator

$$a_k = \sqrt{\frac{1}{2\omega_k \hbar}}\, (ip_k + \mu_k \omega_k x_k)$$

We introduced the Planck constant $\hbar$ only for later convenience of quantization. Then, the free oscillator Hamiltonian in (1) coincides with the energy of the free quantum electromagnetic field (see, e.g., Ford *et al.*, 1988):

$$\frac{1}{8\pi}\int_{\Omega} dX\,((\partial_t \mathbf{C})^2 + (\mathrm{rot}\ \mathbf{C})^2)$$

where in equation (1) (Ford *et al.*, 1988)

$$\mu_k = \frac{4\pi f(k)^2}{\omega_k^2 \Omega}$$

If we restrict ourselves to linear terms in the interaction of the electromagnetic field with the system, then we may choose

$$F_\alpha(x) = \sum_k \omega_k v_{k\alpha} x_k \tag{2}$$

with some unspecified couplings $v_{k\alpha}$ which could be determined if the interaction between the system and the electromagnetic field was explicitly defined in terms of the vector potential C_α. The Hamiltonian interaction (1)–(2) can be obtained from a minimal coupling between a particle and the electromagnetic field if the eigenmodes $g(k, \mathbf{X})$ are expanded in X_α and only the linear terms retained. A relativistic generalization of the model is straightforward; we replace the system Hamiltonian by the Hamiltonian of the quantum scalar field (however, it is still unclear to the author whether dissipation and relativistic invariance are compatible).

When the volume Ω tends to infinity, then the sum over modes has to be replaced by an integral. In such a case we obtain the continuum model

$$H = \frac{1}{2m}\mathbf{P}^2 + V(\mathbf{X}) + \int dk\left(\frac{1}{2\mu_k} p_k^2 + \frac{1}{2}\mu_k\omega_k^2 x_k^2 + \omega_k v_{k\alpha} x_k M_\alpha\right) \tag{3}$$

We shall treat k as one-dimensional for simplicity (the dimension of k is irrelevant).

The Hamiltonian equations of motion in classical mechanics and quantum mechanics (in the Heisenberg picture) with the coupling (2) read

$$\frac{d}{dt}X_\alpha = \frac{1}{m}P_\alpha + \sum_k \omega_k v_{k\beta} x_k \epsilon_{\beta\gamma\alpha} X_\gamma \tag{4}$$

$$\frac{d}{dt}P_\alpha = -\partial_\alpha V + \sum_k \omega_k v_{k\beta} x_k \epsilon_{\beta\gamma\alpha} P_\gamma \tag{5}$$

$$\frac{d}{dt}x_k = \frac{1}{\mu_k} p_k \tag{6}$$

$$\frac{d}{dt} P_k = -\mu_k \omega_k^2 x_k + \omega_k v_{k\beta} \epsilon_{\beta\gamma\alpha} X_\gamma P_\alpha \tag{7}$$

If we assume that $V(\mathbf{X})$ is spherically symmetric and restrict ourselves to the angular momentum M_α as the only relevant variable of the system, then the equations of motion (4)–(7) reduce to

$$\frac{d}{dt} M_\alpha = -\sum_k \omega_k v_{k\beta} x_k \epsilon_{\beta\alpha\gamma} M_\gamma \doteq w_{\alpha\beta}(x) M_\beta \tag{8}$$

$$\frac{d^2}{dt^2} x_k + \omega_k^2 x_k = \frac{\omega_k}{\mu_k} v_{k\alpha} M_\alpha \tag{9}$$

The quantum and classical equations coincide because for the Hamiltonian (1) there is no ordering problem. We can either solve the linear equation (8) for M and insert the solution into equation (9) for x_k or solve the linear equation (9) for x_k and insert its solution into equation (8) for M. Equation (9) says that if we know the behavior of angular momenta $M_\alpha(s)$ as a function of time $s \in [0, t]$ and the initial values of x_k and dx_k/dt, then we can determine the trajectories $x_k(s)$ for $s \in [0, t]$. Conversely, if we know the trajectories $x_k(s)$ for $s \in [0, t]$ and the initial values of M_α, then from equation (8) we can determine $M_\alpha(s)$ in the interval $[0, t]$.

In reality, the initial time is quite arbitrary. We do not know the initial values of the angular momentum whose evolution we want to observe. However, equation (9) suggests that it is sufficient to observe on a time interval $[t_1, t_2]$ the trajectories x_k of the oscillators in order to determine in classical physics the evolution of the angular momenta on the same interval. For this purpose it is sufficient that there exist $u_{k\alpha}$ such that

$$\sum_k \frac{\omega_k}{\mu_k} u_{k\alpha} v_{k\beta} = K\delta_{\alpha\beta} \tag{10}$$

or in the continuum

$$\int dk \frac{\omega_k}{\mu_k} u_{k\alpha} v_{k\beta} = K\delta_{\alpha\beta} \tag{11}$$

where K is a constant. In such a case we can invert equation (9) with the result

$$M_\alpha(t) = K^{-1} \int dk \left(\frac{d^2}{dt^2} x_k + \omega_k^2 x_k\right) u_{k\alpha} \tag{12}$$

From equation (12) we can determine the angular momentum if we know the position and acceleration. In quantum physics $x_k(t)$ are operators which do not commute for different times. Hence, they cannot be measured with arbitrary precision. However, a proper choice of frequencies and the couplings $v_{k\alpha}$ can make the commutator of the integral on the r.h.s. of (12) decay rapidly when the time difference increases (see Section 4 for an example of such a choice). In such a case we could say that the meter consisting of oscillators becomes classical.

Let us consider an example which does not satisfy (11) but nevertheless is useful for a demonstration of the effect of environment observation upon the system's behavior. Assume not only M_3 couples to the environment, i.e.,

$$v_{k\alpha} = \delta_{3\alpha} c_k \tag{13}$$

Then, in addition to $\mathbf{M}^2$ [when $V(\mathbf{X})$ is spherically symmetric, then $\mathbf{M}^2$ is a constant of motion], M_3 is also a constant of motion. Solving (9), we obtain

$$x_k(t) = x_k(0)\cos(\omega_k t) + \frac{p_k(0)}{\mu_k \omega_k}\sin(\omega_k t) + \frac{c_k}{\mu_k \omega_k} M_3(1 - \cos(\omega_k t)) \tag{14}$$

M_α with $\alpha = 1, 2$ satisfy the linear equation

$$\begin{aligned}
&\frac{d}{dt}(M_1 + iM_2)(t) \\
&\quad = -i \sum_k \omega_k c_k \left(\cos(\omega_k t) x_k(0) + \frac{p_k(0)}{\mu_k \omega_k} \sin(\omega_k t) \right) (M_1 + iM_2)(t) \\
&\quad\quad - iM_3 \sum_k \frac{1}{\mu_k} c_k^2 (1 - \cos(\omega_k t))(M_1 + iM_2)(t) \\
&\quad \doteq W(t)(M_1 + iM_2)(t)
\end{aligned} \tag{15}$$

where in the quantum case $x_k(0)$ is the multiplication by x_k, whereas $p_k(0) = -i\hbar\, \partial/\partial x_k$. The commutator of the "frequency operator" W in (15) is

$$[W(t), W(t')] = i\hbar \sum_k \frac{\omega_k c_k^2}{\mu_k} \sin(\omega_k (t - t'))$$

It can decay rapidly for large $|t - t'|$ when we choose the frequencies and couplings properly (see Section 4). We could obtain such a conclusion also for the commutator of the operator $w_{\alpha\beta}(x)$ in (8).

Our suggestion is that through an observation (in the sense of quantum measurement theory) of a collection of quantum oscillators we can gain some information about the dynamics of the angular momentum. Then, through the classical limit of the quantum mechanics of an oscillator (which we consider as well understood) we can understand the process of the measurement in general.

3. SCHRÖDINGER PICTURE: A STOCHASTIC DESCRIPTION

We review in this section the probabilistic formalism of Haba (1994, 1996a). We consider the Hamiltonian

$$H = -\hbar^2 \frac{1}{2m} \Delta + V \tag{16}$$

(we do not distinguish the environment yet). Let U_t ($t \geq 0$) be a unitary Schrödinger evolution determined by this Hamiltonian. We assume that we know $|\chi\rangle_t = U_t|\chi\rangle$. Let us consider an initial wave function of the form $\psi(x) = \chi(x)\phi(x)$, where χ and ϕ are analytic functions. We are interested in the solution ψ_t of the Schrödinger equation with the initial condition ψ,

$$i\hbar\partial_t\psi_t = H\psi_t \tag{17}$$

It can be expressed as $\chi_t\phi_t$, where ϕ_t is the solution of the equation

$$\partial_t\phi_t = \frac{i\hbar}{2m} \Delta\phi_t + \frac{i\hbar}{m} \chi_t^{-1}\nabla\chi_t\nabla\phi_t \tag{18}$$

with the initial condition ϕ.

We express the solution of (18) by a stochastic process (Haba, 1994)

$$(U_t\psi)(x) = \chi_t(x)E[\phi(q_t(x))] \tag{19}$$

where $q_t(x)$ is a complex diffusion process starting at $t = 0$ from x and solving the stochastic differential equation (here $0 \leq \tau \leq t$)

$$dq_\tau = \frac{i\hbar}{m} \chi_{t-\tau}^{-1}\nabla\chi_{t-\tau}(q_\tau)\, d\tau + \lambda\sigma\, db_\tau \tag{20}$$

where

$$\lambda = \frac{1}{\sqrt{2}}(1 + i)$$

and

$$\sigma = \sqrt{\frac{\hbar}{m}}$$

The Brownian motion b_t in equation (20) is defined as the Gaussian process with independent increments and the variance

$$E[b_t^2] = t$$

In order to express the solution of the Schrödinger equation for negative time we can apply the complex conjugation

$$\overline{U_t\overline{\psi}} = \psi_{-t} \tag{21}$$

The action of arbitrary operators on states as well as correlation functions of operators in the Heisenberg picture in arbitrary states can now be expressed as expectation values with respect to the Brownian motion, e.g.,

$$\langle \chi, G_1(x_t)G_2(x)\chi\rangle = E\left[\int dx\, |\chi_t(x)|^2 G_1(x)G_2(q_t(x))\right] \tag{22}$$

Let us consider as an example a model of independent oscillators [the reservoir of equation (1)]

$$H_0 = -\sum_k \frac{\hbar^2}{2\mu_k}\frac{\partial^2}{\partial x_k^2} + \sum_k \frac{1}{2}\mu_k\omega_k^2 x_k^2 \tag{23}$$

The ground-state solution for the Schrödinger equation reads

$$\chi_0(x) = \exp\left(-\sum_k \frac{\mu_k}{2\hbar}\omega_k x_k^2\right) \tag{24}$$

Then, the stochastic equation (20) takes the form

$$dq_k = -i\omega_k q_k\, dt + \lambda\sigma_k\, db_k$$

As an example of a time-dependent solution of the Schrödinger equation let us consider the coherent states $|z\rangle$. The time evolution is

$$\langle x|U_t|z\rangle = \exp\left[-\frac{1}{2\hbar}\sum_k \mu_k\omega_k x_k^2 + \sum_k \frac{\mu_k\omega_k}{\hbar}\exp(-i\omega_k t)z_k x_k + R_t(z)\right] \tag{25}$$

where R is independent of x. Hence, the stochastic process (20) is

$$dq_k = -i\omega_k q_k\, d\tau + i\omega_k \exp[-i\omega_k(t-\tau)]z_k\, d\tau + \lambda\sigma\, db_k \tag{26}$$

The solution of (26) with the initial condition x reads

$$\begin{aligned} q_k(s; z) = {} & \exp(-i\omega_k s)x_k + iz_k \sin(\omega_k s)\exp(-i\omega_k t) \\ & + \lambda\sigma_k \int_0^s \exp[-i\omega_k(s-\tau)]\, db_k(\tau) \end{aligned} \tag{27}$$

4. THE EFFECT OF THE ENVIRONMENT ON THE SCHRÖDINGER EVOLUTION OF THE SYSTEM

We assume that an eigenstate χ_S of the system's Hamiltonian H_S [equation (1) with no environmental oscillators] is known. In order to simplify the notation, we express the eigenstate of H_S in the form $\chi_S = \exp(-S/\hbar)$. When χ_S is the ground state, then the ground-state wave function is positive and ln χ_S is well defined. In our formulas we do not need S itself, but rather [as in equation (20)] $\nabla S = -\hbar\chi_S^{-1}\nabla\chi_S$. In order that (20) make sense, we need to assume that χ_S is a holomorphic function. In such a case the singularities of the gradient of S form a discrete set of poles. The Hamiltonian can be written in the form (from now on we shall denote coordinates of the system by capital letters and the indices of their components by Greek letters; we shall denote by H_S the Hamiltonian of the system normalized by $H_S\chi_S = 0$)

$$H_S = \frac{1}{2m}\sum_{\mu=1}^{n} A_\mu^+ A_\mu - \epsilon \tag{28}$$

where

$$A_\mu = \hbar\frac{\partial}{\partial X_\mu} + \frac{\partial S}{\partial X_\mu}$$

and

$$A_\mu^+ = -\hbar\frac{\partial}{\partial X_\mu} + \frac{\partial S}{\partial X_\mu}$$

These operators satisfy the commutation relations

$$[A_\mu, A_\nu^+] = 2\partial_\mu\partial_\nu S \tag{29}$$

which constitute a generalization of the commutation relations for creation–annihilation operators.

We consider now a coupling (1) of H_S to the angular momentum. It is easy to see that if χ_S is spherically symmetric (e.g., the ground state of a spherically symmetric potential is spherically symmetric), then $\chi_S\chi_0$ [where χ_0 is defined in equation (24)] is an eigenstate of the total Hamiltonian H [equation (1)] with the eigenvalue 0. The stochastic equations (20) applied now to the system and to the environment take the form (we assume the summation convention over the Greek indices)

$$dq_k = -i\omega_k q_k\, ds + \lambda\sigma_k\, db_k \tag{30}$$

$$dQ_\alpha = -\frac{i}{m}\partial_\alpha S(\mathbf{Q})\, ds - \sum_k \nu_{k\beta}\omega_k q_k \epsilon_{\alpha\beta\gamma} Q_\gamma\, ds + \lambda\sigma_m\, db_\alpha \tag{31}$$

[the Brownian motions with different indices in (30)–(31) are independent]. If instead of the ground state χ_0 we take the coherent state, then (31) remains unchanged, but the process $q_k(t)$ is the solution (27) of (26) instead of (30). For general F_α of (1) in (31) we would get the term $\epsilon_{\alpha\beta\gamma}F_\beta(q)Q_\gamma$ instead of the term linear in q.

When we insert the solution $q_k(t)$ of equation (30) into equation (31), we obtain the following closed equation for the system coordinate $\mathbf{Q}$:

$$dQ_\alpha = -\frac{i}{m}\,\partial_\alpha S(\mathbf{Q})\,ds - N_\beta^D(s)\epsilon_{\beta\gamma\alpha}Q_\gamma\,ds - N_\beta^R(s)\epsilon_{\beta\gamma\alpha}Q_\gamma\,ds + \lambda\sigma_m\,db_\alpha \tag{32}$$

where

$$N_\beta^D(s) = \sum_k v_{k\beta}x_k\omega_k \exp(-i\omega_k s) \tag{33}$$

and

$$N_\beta^R(s) = \lambda \sum_k \int_0^s v_{k\beta}\omega_k \exp(-i\omega_k(s-\tau))\sigma_k\,db_k(\tau) \tag{34}$$

correspond to "deterministic" and random noise, respectively.

If instead of the oscillator's ground state χ_0 [equation (24)], we took the coherent state (25), then on the r.h.s. of (32) we would obtain an additional noise term (Haba, 1995)

$$iN_\beta^{\rm coh}(s)\epsilon_{\alpha\beta\gamma}Q_\gamma(s)\,ds$$

where

$$N_\beta^{\rm coh}(s) = i\exp(-i\omega_k t)\sum_k v_{k\beta}z_k\omega_k \sin(\omega_k s) \tag{35}$$

If we are interested in a computation of expectation values (22) in a state $\chi = \chi_0\chi_S$, where χ_0 is defined in (24), then in such computations x_k play the role of independent Gaussian random variables distributed with the density $|\chi_0(x)|^2$ and the covariance (Haba, 1995)

$$E[x_k x_r] = \frac{\hbar}{2\omega_k\mu_k}\,\delta_{kr} \tag{36}$$

Hence, the noise N_α^D can be treated as a complex stochastic process with the covariance

$$E_D[N_\alpha^D(s)N_\beta^D(\tau)] = \frac{\hbar}{2}\sum_k \frac{\omega_k}{\mu_k}\,v_{k\alpha}v_{k\beta}\exp(-i\omega_k(\tau+s)) \tag{37}$$

$$E_D[\overline{N_\alpha^D(s)}N_\beta^D(\tau)] = \frac{\hbar}{2}\sum_k \frac{\omega_k}{\mu_k}\,v_{k\alpha}v_{k\beta}\exp(-i\omega_k(\tau-s))$$

where $E_D[\,\cdot\,]$ means that the average is taken only over the initial values of the coordinates of the environment.

Next, we find through a direct calculation

$$E[N^R_\alpha(s)N^R_\beta(\tau)]$$
$$= \frac{\hbar}{2}\sum_k \frac{\omega_k}{\mu_k} v_{k\alpha}v_{k\beta}[\exp(-i\omega_k|\tau - s|) - \exp(-i\omega_k(\tau + s))] \tag{38}$$

There is a large freedom in the choice of frequencies and couplings. However, this arbitrariness can be limited by the requirement that the environment behaves like a classical reservoir (more precisely, like classical white noise). We assume that the sum over frequencies in (37) can be approximated by the integral

$$\int_0^\infty dk\, v_{\alpha k}v_{\beta k}\frac{\omega_k}{\mu_k}\exp(-ik\omega s) = 2ag_{\alpha\beta}\left(\delta(s) - \frac{1}{\pi s}\right)$$

where $g_{\alpha\beta}$ is a certain positive-definite constant tensor and a is a certain positive parameter. This behavior of the sum means that $\omega_k \sim \omega k$ and

$$v_{k\alpha}v_{k\beta}\frac{\omega_k}{\mu_k} \simeq ag_{\alpha\beta}$$

Then, the term s^{-1} is considered as small in comparison to $\delta(s)$. Under these assumptions on $v_{k\mu}$ and ω_k the quickly oscillating terms in (37) are negligible and we obtain an approximation

$$E_D[N^D(s)N^D(\tau)] = 0$$
$$E_D[\overline{N^D_\alpha(s)}N^D_\beta(\tau)] = \hbar ag_{\alpha\beta}\delta(\tau - s) \tag{39}$$

We define the complex Brownian motion $\mathbf{B}^D$ by

$$e_{\alpha\beta}\sqrt{a\hbar}\frac{dB^D_\beta}{ds} = N^D_\alpha$$

where the matrix e is the square root of g, i.e.,

$$e_{\alpha\nu}e_{\beta\nu} = g_{\alpha\beta}$$

Under our assumptions on $v_{k\alpha}$ from equation (38) we obtain

$$E[N^R_\alpha(s)N^R_\beta(\tau)] = \hbar ag_{\alpha\beta}\delta(s - \tau) \tag{40}$$

[the second term on the r.h.s. of (38), as quickly oscillating, is negligible]. If we introduce the real Brownian motion as a realization of the noise, i.e.,

$$e_{\alpha\beta}\sqrt{\hbar a}\frac{dB^R_\beta}{ds} = N^R_\alpha$$

then we obtain the following mathematical version of equation (32):

$$dQ_\alpha = -\frac{i}{m}\,\partial_\alpha S(\mathbf{Q})\,ds - \sqrt{a\hbar}\,\epsilon_{\alpha\beta\gamma}Q_\gamma e_{\beta\nu}\,dB_\nu^D \\ - \sqrt{a\hbar}\,\epsilon_{\alpha\beta\gamma}Q_\gamma e_{\beta\nu}\,dB_\nu^R + \lambda\sigma_m\,db_\alpha \tag{41}$$

where B^D, B^R, and b are independent Brownian motions.

5. OBSERVATION OF OSCILLATOR'S COORDINATES

A description of the position measurement results already from Born's interpretation of the wave function. In this sense it is the minimal addition to the Schrödinger equation. We investigate in this section the consequences of a position measurement performed upon the environment on the posterior evolution of the system.

First, consider an initial state $\Psi(x, \mathbf{X}) \doteq \chi_0(x)\chi_S(\mathbf{X})\phi(\mathbf{X})$ which is a product of the ground state of the environment and an arbitrary state of the system. It evolves in time from $\tau = 0$ to $\tau = t$. At time t a measurement of positions x_k of the oscillators is performed, but no account of the results is taken (a nonselective measurement). In such a case, according to standard quantum mechanics, after this measurement the initial pure state Ψ is transformed into a mixed state ρ_P,

$$|\Psi_t\rangle\langle\Psi_t| \to \rho_P(t) = \sum_r P_r|\Psi_t\rangle\langle\Psi_t|P_r \tag{42}$$

where P_r denotes a projection onto the regions of the oscillator's configuration space where the oscillator coordinates are looked at. If the measurement is complete and exhaustive, i.e., runs over small surroundings of all the points of the configuration space, then the sum can be replaced by an integral

$$\rho_P(t; \mathbf{X}, \mathbf{X}') \\ = \int \prod_k dx_k\, \overline{\chi_S(\mathbf{X})}\chi_S(\mathbf{X}')|\chi_0(x)|^2\overline{E[\phi(\mathbf{Q}_t(\mathbf{X}, x))]}E[\phi(\mathbf{Q}_t(\mathbf{X}', x))] \tag{43}$$

For the model (1) the stochastic process $\mathbf{Q}_t(\mathbf{X}, x)$ is the solution of equation (31) with the initial conditions $\mathbf{X}$ for $\mathbf{Q}$ and x_k for q_k. When we apply the definition of the noise N^D [equation (39)], then the integration over x_k can be expressed as an average over N^D. In fact, after the approximations performed in Section 4 to arrive at (41), we treat from now on (41) as our idealized mathematical model of the interaction of a system with the environment described otherwise as an infinite-dimensional Hamiltonian system (1). So, as a consequence of (41), we obtain the following formula for the evolution of the density matrix:

$$\rho_P(t; \mathbf{X}, \mathbf{X}') = \overline{\chi_S(\mathbf{X})}\chi_S(\mathbf{X}')E_D[\overline{E[\phi(\mathbf{Q}_t(\mathbf{X}))]}E[\phi(\mathbf{Q}_t(\mathbf{X}'))]] \tag{44}$$

where $E[\cdot]$ means the expectation value with respect to the Brownian motions $\mathbf{b}$ and $\mathbf{B}^R$, whereas E_D means the expectation value with respect to the complex Brownian motion $\mathbf{B}^D$. An application of the standard Ito stochastic calculus (Ikeda and Watanabe, 1981) shows that $\phi(\mathbf{Q}_t)$ fulfills the Ito equation (M denotes the angular momentum)

$$\begin{aligned} d\phi = &-\frac{i}{m}\frac{\partial\phi}{\partial Q_\alpha}\partial_\alpha S\, ds - \frac{i}{\hbar}\sqrt{a\hbar}e_{\alpha\nu}M_\nu\phi\, dB_\alpha^D \\ &- \frac{i}{\hbar}\sqrt{a\hbar}e_{\alpha\nu}M_\nu\phi\, dB_\alpha^R + \lambda\sigma_m\partial_\alpha\phi\, db_\alpha - \frac{a}{2\hbar}g_{\alpha\beta}M_\alpha M_\beta\phi\, ds \\ &+ i\frac{\hbar}{2m}\Delta\phi\, ds \end{aligned} \tag{45}$$

This equation can be considered as a stochastic perturbation of the Schrödinger equation because at $a = 0$ on the r.h.s. of (45) the terms without the Ito differentials have the form $\tilde{H}_S\phi\, ds$, where $\tilde{H}_S = \chi_S^{-1}H_S\chi_S$. We can rewrite (45) in terms of $\psi(\mathbf{Q}_t) = \chi_S(\mathbf{Q}_t)\phi(\mathbf{Q}_t)$. It satisfies the stochastic Ito–Schrödinger equation

$$\begin{aligned} &\chi(\mathbf{Q}_t)d(\chi(\mathbf{Q}_t)^{-1}\psi(\mathbf{Q}_t)) \\ &\quad = -\frac{i}{\hbar}H_S\psi(\mathbf{Q})\, ds - \frac{i}{\hbar}\sqrt{a\hbar}e_{\alpha\nu}M_\nu\psi(\mathbf{Q})\, db_\alpha^D \\ &\qquad - \frac{i}{\hbar}\sqrt{a\hbar}e_{\alpha\nu}M_\nu\psi(\mathbf{Q})\, dB_\alpha^R + \lambda\sigma_m\partial_\alpha\psi(\mathbf{Q})\, db_\alpha - \frac{a}{2\hbar}g_{\alpha\beta}M_\alpha M_\beta\psi(\mathbf{Q})\, ds \end{aligned} \tag{46}$$

If we normalized each sample path $\psi(\mathbf{Q}_t)$, we would obtain the nonlinear Ito–Schrödinger equation of Gisin (1984) and Gisin and Percival (1992). This is not surprising, because the form of such an equation is determined by the requirement that it should be a stochastic perturbation of the Schrödinger equation preserving the norm of the random vector ψ. There is a minor distinction between (46) and the equation of Gisin (1984) and Gisin and Percival (1992). It comes from the transformation $\psi \to \chi^{-1}\psi$, which is time dependent because the process $\mathbf{Q}$ is time dependent. For this reason we have obtained the modified time differential on the l.h.s. of (46). In order to derive the master equation for ρ_P we differentiate ρ_P in (44) and apply (45). Then, elementary rules of the stochastic calculus (Ikeda and Watanabe, 1981) lead to the formula [the linear and nonlinear forms of the Ito–Schrödinger equations lead to the same master equations]

$$\partial_t\rho_P = -\frac{i}{\hbar}[H_S, \rho_P] - \frac{a}{2\hbar}(g_{\alpha\beta}M_\alpha M_\beta\rho_P + \rho_P g_{\alpha\beta}M_\alpha M_\beta) + \frac{a}{\hbar}g_{\alpha\beta}M_\alpha\rho_P M_\beta \tag{47}$$

We have obtained the master equation in the Lindblad form (Lindblad, 1976; Gorini *et al.*, 1976). The approximation which leads to the stochastic equation (41) is a Markovian approximation implying linear differential equations for the density matrix. The Lindblad form of the evolution equation ensures that the evolution preserves the normalization of the density matrix and its positivity. The solution of (47) defines a semigroup. From the semigroup composition law it follows that the subsequent evolution of $\rho P(t)$ does not depend on the time $s \leq t$ when the measurement was performed.

Let us consider as an example the model (13). Then,

$$g_{\alpha\beta} = \delta_{3\alpha}\delta_{3\beta}$$

We have assumed that the potential is spherically symmetric. In such a case $\mathbf{M}^2$ is a constant of motion. Hence, we can choose a basis $|n, l, m_3\rangle$ where $H_S|n, l, m_3\rangle = \epsilon_{nl}|n, l, m_3\rangle$, $\mathbf{M}^2|n, l, m_3\rangle = \hbar^2 l(l + 1)|n, l, m_3\rangle$, and $M_3|n, l, m_3\rangle = \hbar\, m_3|n, l, m_3\rangle$. In this basis the master equation (47) reads

$$\partial_t \rho_{nlm_3,kjm_3'} = -\frac{i}{\hbar}(\epsilon_{nl} - \epsilon_{kj})\rho_{nlm_3,kjm_3'} - \frac{a\hbar}{2}(m_3 - m_3')^2 \rho_{nlm_3,kjm_3'} \tag{48}$$

It follows from this equation that for a large time

$$\rho_{nlm_3,kjm_3'} \simeq \exp\left(-\frac{a\hbar}{2}(m_3 - m_3')^2 t\right) \tag{49}$$

In the terminology of Zurek (1981, 1982; Unruh and Zurek, 1989), the eigenstates of M_3 constitute the pointer basis in our model (1).

6. DISCUSSION AND OUTLOOK

Stochastic equations of the form (45)–(46) (and their nonlinear version resulting from a normalization of the random vector) have been postulated in Gisin (1984) and Ghirardi *et al.* (1990). It has been shown in Caves and Milburn (1987) that the continuous (in time) measurement of a system's position leads to a master equation of the form of that of Ghirardi *et al.* (1990). We have derived a stochastic Schrödinger equation and a corresponding master equation related to the energy measurement (assuming an interaction with an environment) in Haba (1996b). Barchielli *et al.* (1982) suggested that the master equation for a measurement of any observable results from standard quantum mechanics if the measurement is repeated in time and subsequently a continuum limit is taken. A stochastic equation resembling our equation (45) is suggested in Gisin (1989), Gisin and Cibils (1992), and Amman (1994). In Sanders and Milburn (1989) and Milburn (1988) a master equation of the type (47) is derived under the assumption of a continuous angular momentum measurement. In this paper we have shown that a stochastic equation for a state vector results as a posterior evolution equation after a nonselective measurement of positions of particles of the environment. If we make a selective measurement upon the environment, i.e., the sum over P_r in (42) is finite, e.g., P_r is a projection operator onto the region

$$|x_k| \leq a_k \tag{50}$$

then the integration over x in (43) becomes bounded. If we still insist upon the expression of the time evolution by the Brownian motion $\mathbf{B}^D$, then the sample paths of the Brownian motion are restricted to the region (50) [in accordance with the Fourier expansion (33) of the sample paths].

In this paper we discussed only the measurement of oscillator positions in the oscillator ground state. We have indicated how the ground state could be replaced by a coherent state. Then we could treat the simultaneous approximate measurement of position and momentum as well as their classical limit. As indicated in Section 3, working with the coherent state is equivalent to the introduction of an additional noise N^{coh} [equation (35)]. When a statistical distribution of the variables z_k (position and momentum) is introduced, then we can treat a measurement at finite temperature and investigate the effect of thermal noise on the quantum measurement.

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